Computational methods for several models of ice stream flow

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Bathymetry and stickyness distribution

- Bathymetry:
  - Aspect ratio $\varepsilon = [H]/[x] \ll 1$
  - Need surface *and* bed slopes to be small
- Stickyness distribution:
  - Limiting cases of plug flow versus vertical shear
  - Stress ratio: $\lambda = [\tau_{xz}]/[\tau_{\text{membrane}}]$
  - Discontinuous: frozen to slippery transition at ice stream margins
- Bed slope is discontinuous and of order 1.
  - Taylor expansions no longer valid
  - high resolution, subgrid parametrization, short time steps

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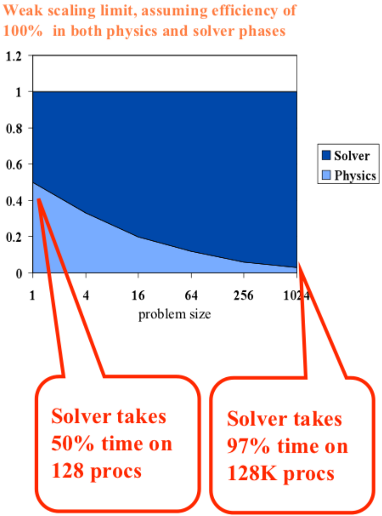
It's *all* about algorithms (at the petascale)

- **Given, for example:**
  - a "physics" phase that scales as $O(N)$
  - a "solver" phase that scales as $O(N^{3/2})$
  - computation is almost all solver after several doublings
- **Most applications groups have not yet "felt" this curve in their gut**
  - as users actually get into queues with more than 4K processors, this will change



(c/o David Keyes)

Hydrostatic equations for ice sheet flow

- Valid when $w_x \ll u_z$, independent of basal friction (Schoof&Hindmarsh 2010)
- Eliminate p and w from Stokes by incompressibility:
  3D elliptic system for $u = (u, v)$

$$-\nabla \cdot \left[\eta \begin{pmatrix} 4u_x + 2v_y & u_y + v_x & u_z \\ u_y + v_x & 2u_x + 4v_y & v_z \end{pmatrix} \right] + \rho g \overline{\nabla} h = 0$$

$$\eta(\theta, \gamma) = \frac{B(\theta)}{2} (\gamma_0 + \gamma)^{\frac{1-\mathfrak{n}}{2\mathfrak{n}}}, \qquad \mathfrak{n} \approx 3$$

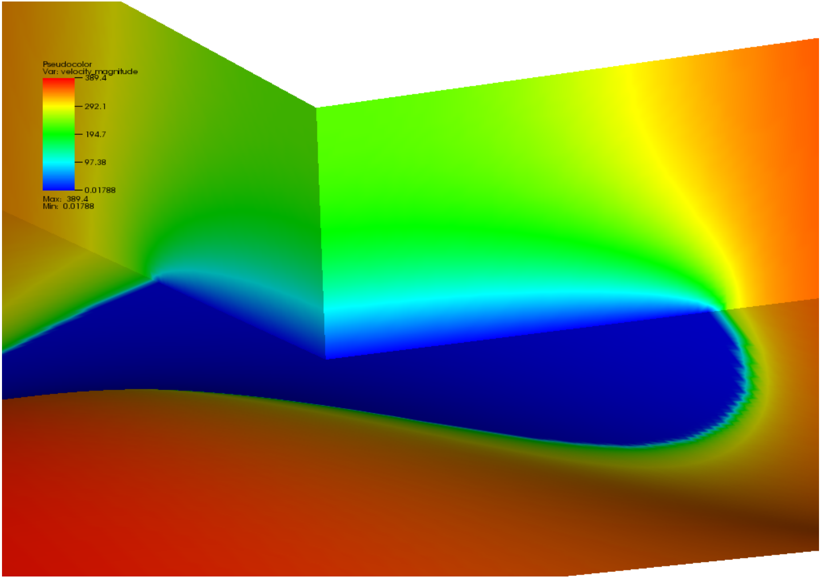
$$\gamma = u_x^2 + v_y^2 + u_x v_y + \frac{1}{4}(u_y + v_x)^2 + \frac{1}{4} u_z^2 + \frac{1}{4} v_z^2$$

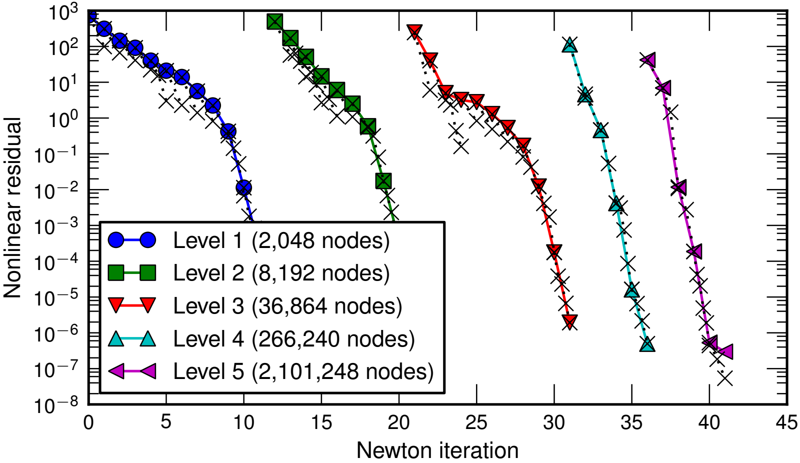
and slip boundary $\sigma \cdot n = \beta^2 u$ where

$$\beta^2(\gamma_b) = \beta_0^2 (\varepsilon_b^2 + \gamma_b)^{\frac{\mathfrak{m}-1}{2}}, \qquad 0 < \mathfrak{m} \leq 1$$

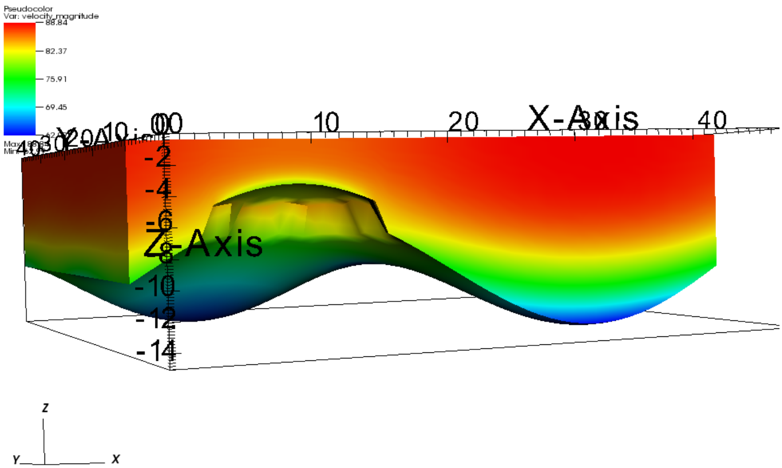
$$\gamma_b = \frac{1}{2}(u^2 + v^2)$$

- Q_1 FEM with Newton-Krylov-Multigrid solver in PETSc: `src/snes/examples/tutorials/ex48.c`


Pseudocolor
Var: velocity_magnitude
389.4
292.1
194.7
97.38
0.01788
Max: 389.4
Min: 0.01788



Grid-sequenced Newton-Krylov solution of test X. The solid lines denote nonlinear iterations, and the dotted lines with $\times$ denote linear residuals.



- Bathymetry is essentially discontinuous on any grid
- Shallow ice approximation produces oscillatory solutions
- Nonlinear and linear solvers have major problems or fail
- Grid sequenced Newton-Krylov multigrid works
  as well as in the smooth case

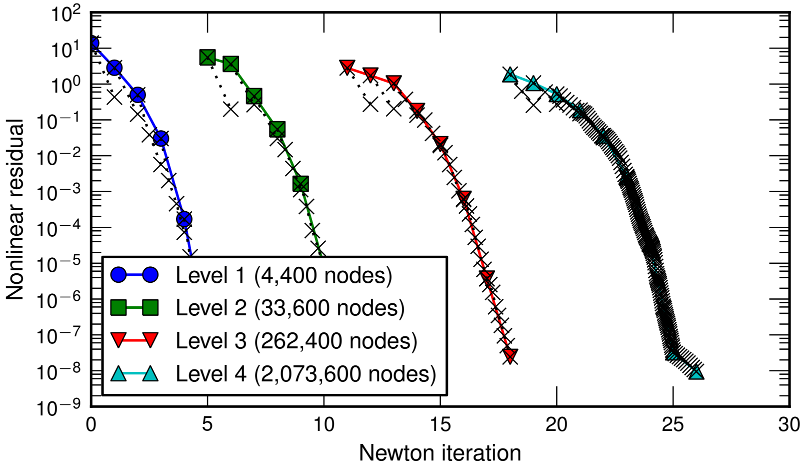


Figure: Grid sequenced Newton-Krylov convergence for test Y. The "cliff" has 58° angle in the red line (12×125 meter elements), 73° for the cyan line (6×62 meter elements).

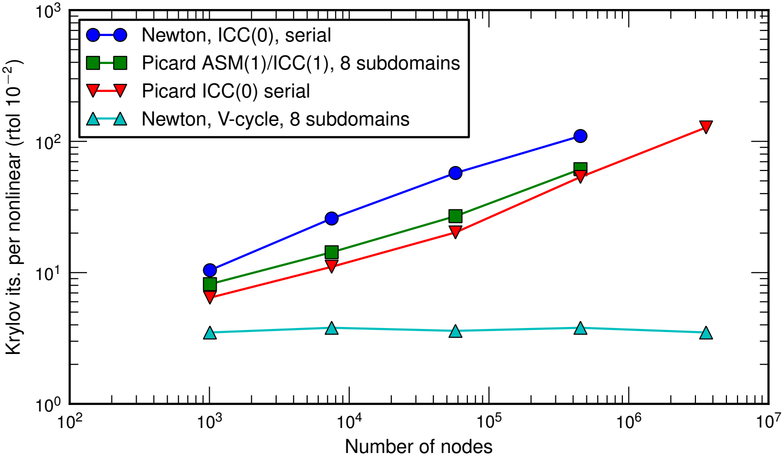
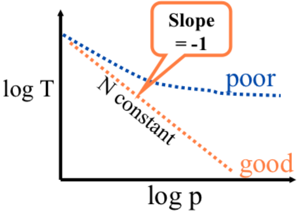


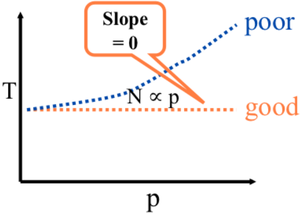
Figure: Average number of Krylov iterations per nonlinear iteration. Each nonlinear system was solved to a relative tolerance of 10^{-2}.

Review: two definitions of scalability

- **"Strong scaling"**
  - **execution time (*T*) decreases in inverse proportion to the number of processors (*p*)**
  - ***fixed size problem* (*N*) *overall***
  - **often instead graphed as reciprocal, "speedup"**



- **"Weak scaling" (memory bound)**
  - **execution time remains constant, as problem size and processor number are increased in proportion**
  - ***fixed size problem per processor***
  - **also known as "Gustafson scaling"**



(c/o David Keyes)

Strong scaling on Blue Gene/P (Shaheen)

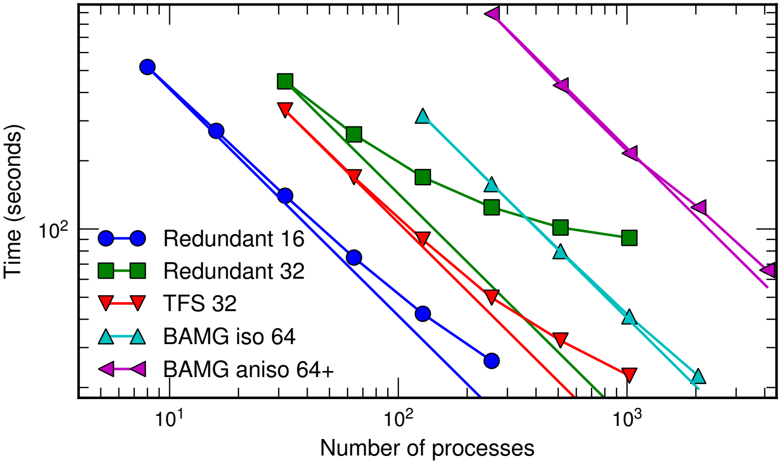


Figure: Strong scaling on Shaheen for different size coarse levels problems and different coarse level solvers. The straight lines on the strong scaling plot have slope -1 which is optimal.

Weak scaling on Blue Gene/P (Shaheen)

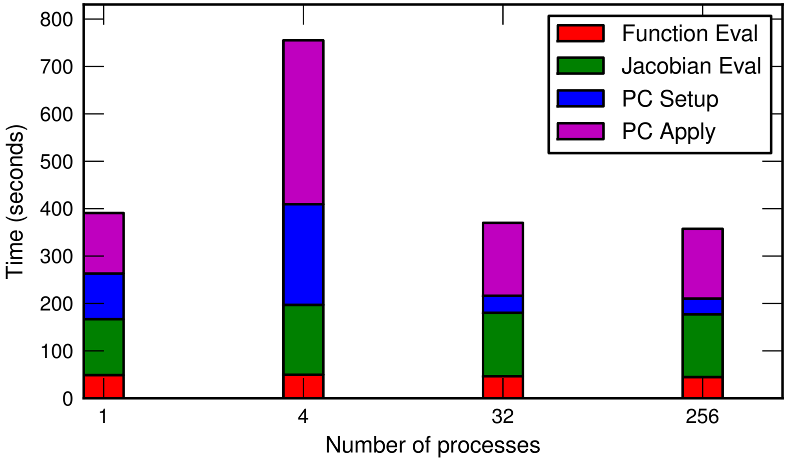


Figure: Weak scaling on Shaheen with a breakdown of time spent in different phases of the solution process. Times are for the full grid-sequenced problem, not just the finest level solve.

One high-accuracy solve costs 30 times as much as a residual evaluation

about 15 to reach truncation error

1000 times faster than existing methods

(Brown, Smith, Ahmadia 2011; submitted to JGR)

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Standard shallow approximations

- Shallow Ice Approximation (SIA):
  - Purely local definition of velocity

$$u(z) = -(\rho g)^{\mathfrak{n}} \int_b^z A(T(z))(h-z)^{\mathfrak{n}} \left|\overline{\nabla} h\right|^{\mathfrak{n}-1} \overline{\nabla} h$$

  - No membrane stresses so only acceptable for $\lambda \gg 1$
  - No solve so costs similar to one residual evaluation per time step
- "Shelfy Stream" Approximation (SSA)
  - Need to solve elliptic problem posed in the map plane (2D):

$$-\overline{\nabla} \cdot \left[H\eta \begin{pmatrix} 4\overline{u}_x + 2\overline{v}_y & \overline{v}_x + \overline{u}_y \\ \overline{v}_x + \overline{u}_y & 2\overline{u}_x + 4\overline{v}_y \end{pmatrix} \right] + \beta^2 \overline{u} + \rho g H \overline{\nabla} h = 0$$

$$\eta(\theta, \gamma) = \frac{B(\theta)}{2} (\gamma_0 + \gamma)^{\frac{1-\mathfrak{n}}{2\mathfrak{n}}}, \qquad \mathfrak{n} \approx 3$$

$$\gamma = \overline{u}_x^2 + \overline{v}_y^2 + \overline{u}_x \overline{v}_y + \frac{1}{4}(\overline{u}_y + \overline{v}_x)^2$$

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Vertically-integrated Hybrids

- Daniel Goldberg 2010, same order of accuracy as hydrostatic
  - Vertically average "membrane" part of hydrostatic equations

$$-\overline{\nabla}\cdot\left[\overline{\eta}\begin{pmatrix}4\overline{u}_x+2\overline{v}_y & \overline{v}_x+\overline{u}_y\\ \overline{v}_x+\overline{u}_y & 2\overline{u}_x+4\overline{v}_y\end{pmatrix}\right]-\left[\eta\begin{pmatrix}u_z\\ v_z\end{pmatrix}\right]_z+\beta^2\overline{u}+\rho g H\overline{\nabla}h=0$$

$$\eta(\theta,\gamma)=\frac{B(\theta)}{2}(\gamma_0+\gamma)^{\frac{1-\mathfrak{n}}{2\mathfrak{n}}},\qquad \mathfrak{n}\approx 3$$

$$\gamma=\overline{u}_x^2+\overline{v}_y^2+\overline{u}_x\overline{v}_y+\frac{1}{4}(\overline{u}_y+\overline{v}_x)^2+\frac{1}{4}u_z^2+\frac{1}{4}v_z^2$$

  - Solve by integrating z-dependence and η, solve linear elliptic problem in map plane for $\overline{u}$, iterate (Picard, ≈ 50 its)
  - Evaluating viscosity (or a Newton residual) costs about one hydrostatic residual
- Bueler and Brown 2009, used in PISM
  - Ad-hoc combination of independent SSA and SIA solutions
  - Lower formal order of accuracy, but nonlinear solve is strictly 2D

Non-Newtonian Stokes system: velocity u, pressure p

$$-\nabla\cdot(\eta Du) + \nabla p - f = 0$$
$$\nabla\cdot u = 0$$

$$Du = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right)$$
$$\gamma(Du) = \frac{1}{2} Du : Du$$
$$\eta(\gamma) = B(\theta,\dots)(\gamma_0+\gamma)^{\frac{\mathfrak{p}-2}{2}}$$
$$\mathfrak{p} = 1 + \frac{1}{\mathfrak{n}} \approx \frac{4}{3}$$
$$T = 1 - n\otimes n$$

with boundary conditions

$$(\eta Du - p1)\cdot n = \begin{cases} 0 & \text{free surface} \\ -\rho_w z n & \text{ice-ocean interface} \end{cases}$$
$$u = 0 \qquad \text{frozen bed}, \theta < \theta_0$$
$$\left.\begin{aligned} u\cdot n &= g_{\text{melt}}(Tu,\dots) \\ T(\eta Du - p1)\cdot n &= g_{\text{slip}}(Tu,\dots) \end{aligned}\right\} \text{nonlinear slip}, \theta \ge \theta_0$$
$$g_{\text{slip}}(Tu) = \beta_{\mathfrak{m}}(\dots)|Tu|^{\mathfrak{m}-1}Tu$$

Navier $\mathfrak{m} = 1$, Weertman $\mathfrak{m} \approx \frac{1}{3}$, Coulomb $\mathfrak{m} = 0$.

Stokes challenges

Mass conservation is critical

- Staggered grid finite difference (hard to deal with geometry)
- Stabilized methods (conservation artifacts when non-smooth)
- Inf-sup stable mixed finite element method
  - Use discontinuous pressure to enforce local mass conservation
  - Inf-sup constant decays like $\sqrt{\varepsilon}$ for $Q_k - P_{k-1}^{\text{disc}}$
  - Sub-optimal order of accuracy for $Q_k - Q_{k-2}^{\text{disc}}$

Solving saddle-point problems

- Not uniformly elliptic: solvers are much less robust
- Standard preconditioners do not work
- Coupled multigrid with Vanka smoothers offer best performance, not robust for stretched grids or anisotropic viscosity
- Block preconditioners require approximate commutators, fragile for strong anisotropy and non-smooth viscosity

Outlook

- We have textbook multigrid efficiency for hydrostatic equations
- All other models are currently slower at high resolution
  because there are no scalable implementations
- Daniel Goldberg's model could be as much as 4 times faster,
  probably closer to 2 times
- Bueler & Brown's model (or SSA) could be up to 10 times faster
- Technical challenges for Stokes
- Bathymetry is rough enough that we should solve Stokes
- Singularities: reentrant corners, transition from frozen
  to slip bounadry conditions, grounded margins, grounding lines
- Implicit time integration