Implicit solution of free surface flows in glaciology

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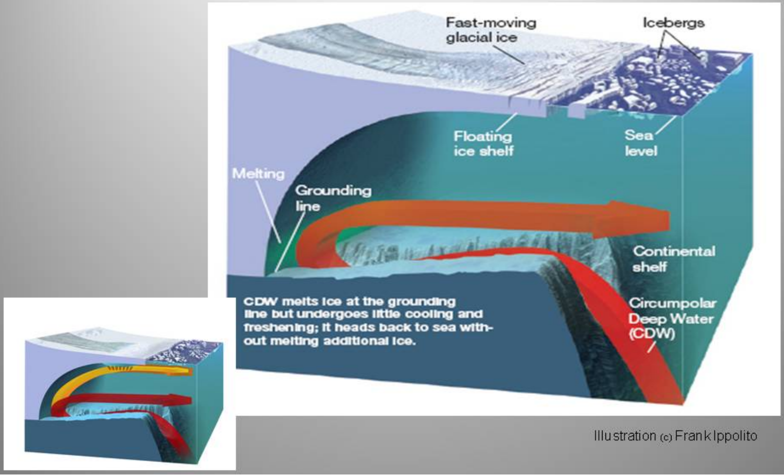
Antarctic Ocean-Ice Interaction



Bindschadler 2008

Hydrostatic equations for ice sheet flow

- Valid when $w_x \ll u_z$, independent of basal friction (Schoof&Hindmarsh 2010)
- Eliminate p and w from Stokes by incompressibility:
  3D elliptic system for $u = (u, v)$

$$-\nabla \cdot \left[\eta \begin{pmatrix} 4u_x + 2v_y & u_y + v_x & u_z \\ u_y + v_x & 2u_x + 4v_y & v_z \end{pmatrix} \right] + \rho g \overline{\nabla} h = 0$$

$$\eta(\theta, \gamma) = \frac{B(\theta)}{2} (\gamma_0 + \gamma)^{\frac{1-\mathfrak{n}}{2\mathfrak{n}}}, \qquad \mathfrak{n} \approx 3$$

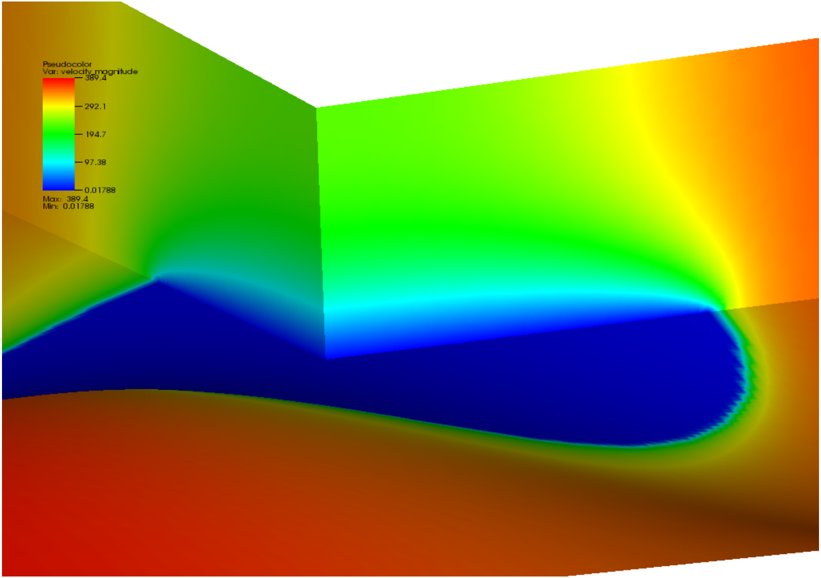
$$\gamma = u_x^2 + v_y^2 + u_x v_y + \frac{1}{4}(u_y + v_x)^2 + \frac{1}{4}u_z^2 + \frac{1}{4}v_z^2$$

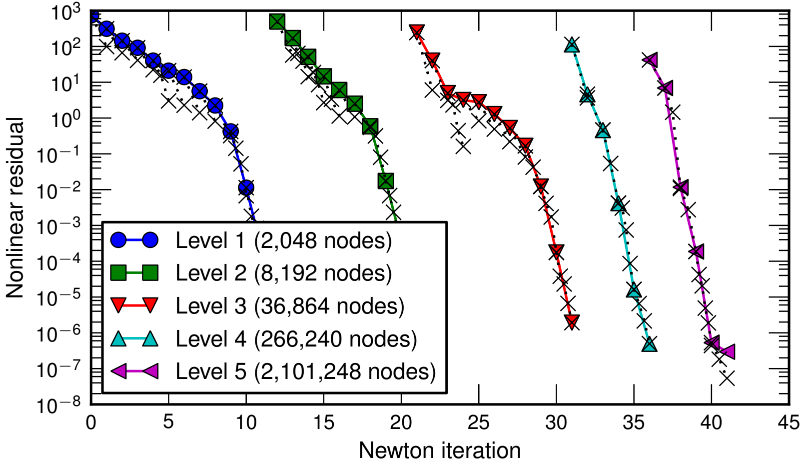
and slip boundary $\sigma \cdot n = \beta^2 u$ where

$$\beta^2(\gamma_b) = \beta_0^2 (\varepsilon_b^2 + \gamma_b)^{\frac{\mathfrak{m}-1}{2}}, \qquad 0 < \mathfrak{m} \le 1$$

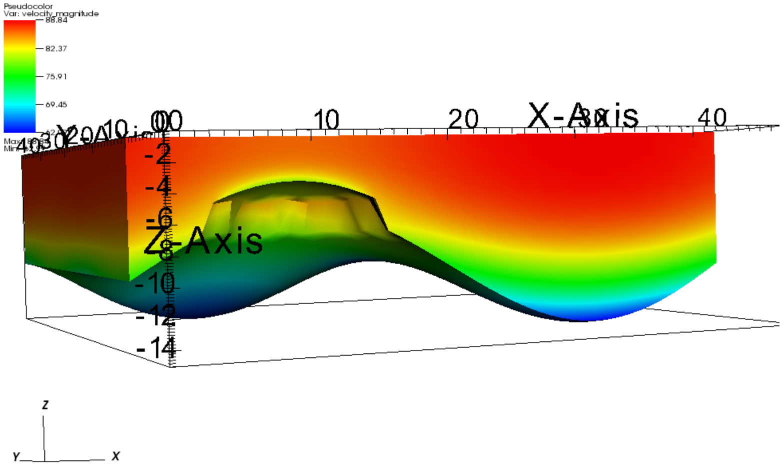
$$\gamma_b = \frac{1}{2}(u^2 + v^2)$$

- Q_1 FEM with Newton-Krylov-Multigrid solver in PETSc:
  `src/snes/examples/tutorials/ex48.c`


Pseudocolor
Var: velocity_magnitude
389.4
292.1
194.7
97.38
0.01788
Max: 389.4
Min: 0.01788



Grid-sequenced Newton-Krylov solution of test X. The solid lines denote nonlinear iterations, and the dotted lines with $\times$ denote linear residuals.



- Bathymetry is essentially discontinuous on any grid
- Shallow ice approximation produces oscillatory solutions
- Nonlinear and linear solvers have major problems or fail
- Grid sequenced Newton-Krylov multigrid works
  as well as in the smooth case

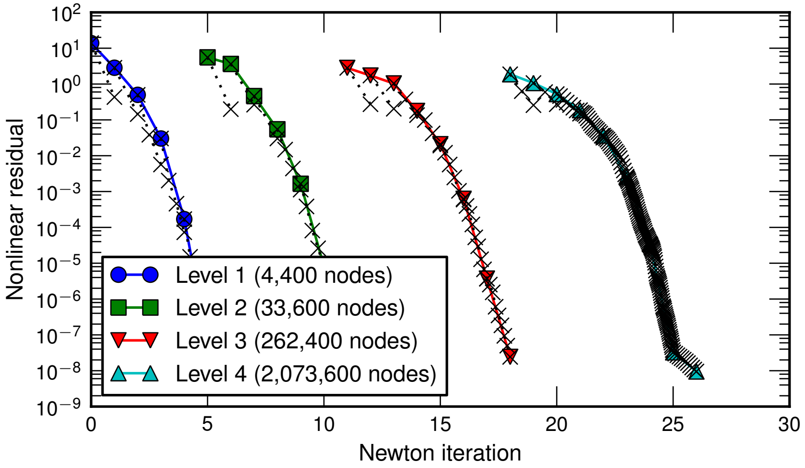


Figure: Grid sequenced Newton-Krylov convergence for test Y. The "cliff" has 58° angle in the red line (12×125 meter elements), 73° for the cyan line (6×62 meter elements).

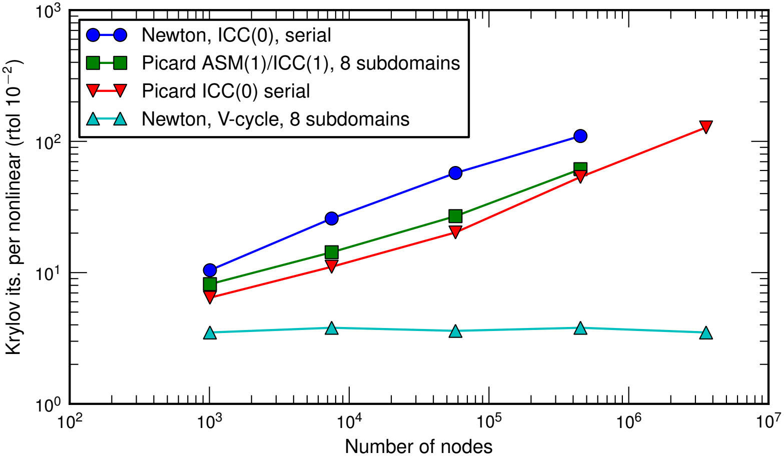


Figure: Average number of Krylov iterations per nonlinear iteration. Each nonlinear system was solved to a relative tolerance of 10^{-2}.

Strong scaling on Blue Gene/P (Shaheen)

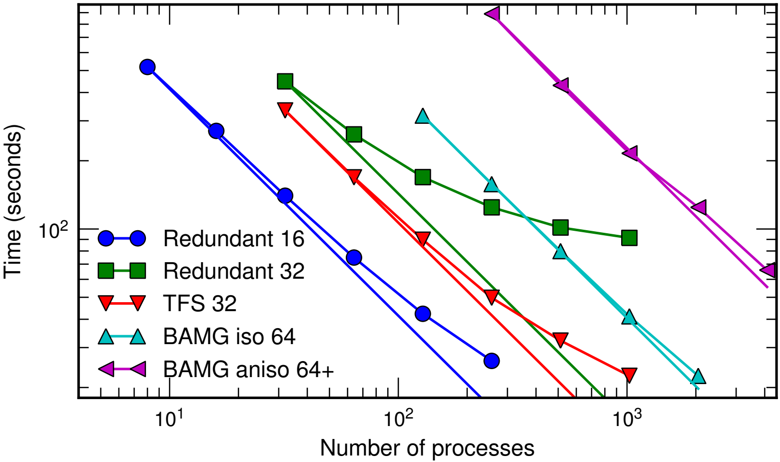


Figure: Strong scaling on Shaheen for different size coarse levels problems and different coarse level solvers. The straight lines on the strong scaling plot have slope -1 which is optimal.

Weak scaling on Blue Gene/P (Shaheen)

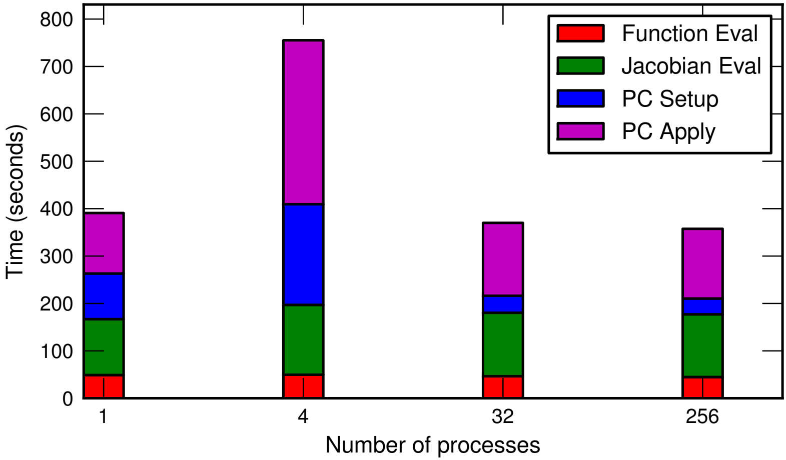
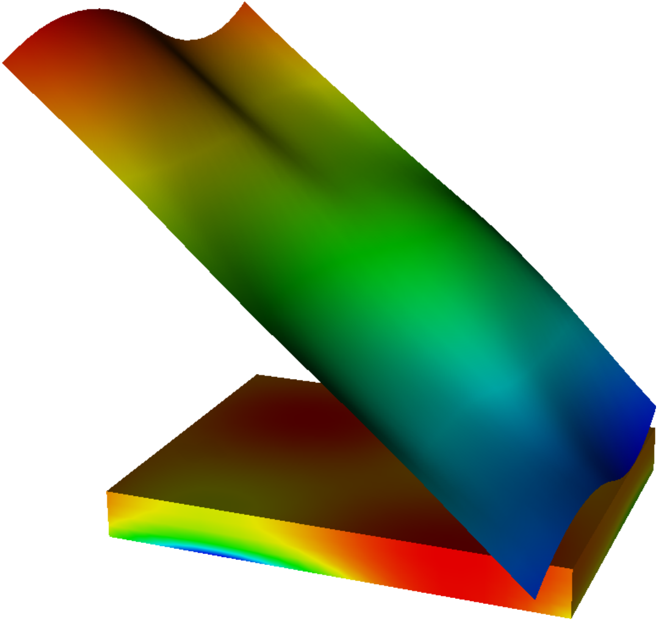


Figure: Weak scaling on Shaheen with a breakdown of time spent in different phases of the solution process. Times are for the full grid-sequenced problem, not just the finest level solve.



Non-Newtonian Stokes system: velocity u, pressure p

$$-\nabla \cdot (\eta Du) + \nabla p - f = 0$$
$$\nabla \cdot u = 0$$

$$Du = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right)$$
$$\gamma(Du) = \frac{1}{2} Du : Du$$
$$\eta(\gamma) = B(\Theta, \dots)\left(\gamma_0 + \gamma\right)^{\frac{\mathfrak{p}-2}{2}}$$
$$\mathfrak{p} = 1 + \frac{1}{\mathfrak{n}} \approx \frac{4}{3}$$
$$T = 1 - n \otimes n$$

with boundary conditions

$$(\eta Du - p1) \cdot n = \begin{cases} 0 & \text{free surface} \\ -\rho_w z n & \text{ice-ocean interface} \end{cases}$$
$$u = 0 \qquad \text{frozen bed}, \Theta < \Theta_0$$
$$\left.\begin{aligned} u \cdot n &= g_{\text{melt}}(Tu, \dots) \\ T(\eta Du - p1) \cdot n &= g_{\text{slip}}(Tu, \dots) \end{aligned}\right\} \text{nonlinear slip}, \Theta \geq \Theta_0$$
$$g_{\text{slip}}(Tu) = \beta_{\mathfrak{m}}(\dots)|Tu|^{\mathfrak{m}-1} Tu$$

Navier $\mathfrak{m} = 1$, Weertman $\mathfrak{m} \approx \frac{1}{3}$, Coulomb $\mathfrak{m} = 0$.

Other critical equations

- Mesh motion: x

$$-\nabla\cdot\sigma = 0 \qquad \sigma = \mu\left[2Dw + (\nabla w)^T\nabla w\right] + \lambda|\nabla w|1$$

surface: $(\dot{x}-u)\cdot n = q_{BL},\ T\sigma\cdot n = 0$ $\qquad w = x - x_0$

- Heat transport: Θ (enthalpy)

$$\frac{\partial}{\partial t}\Theta + (u-\dot{x})\cdot\nabla\Theta$$
$$-\nabla\cdot\left[\kappa_T(\Theta)\nabla T(\Theta) + \kappa_\omega\nabla\omega(\Theta) + q_D(\Theta)\right] - \eta Du:Du = 0$$

  - ALE advection
  - Thermal diffusion
  - Moisture diffusion/Darcy flow
  - Strain heating

Note: $\kappa(\Theta)$ and $q_D(\Theta)$ are very sensitive near $\Theta = \Theta_0$

Summary of primal variables in DAE

u	velocity	algebraic
p	pressure	algebraic
x	mesh location	algebraic in domain, differential at surface
Θ	enthalpy	differential

Stokes challenges

Mass conservation is critical

- Staggered grid finite difference (hard to deal with geometry)
- Stabilized methods (conservation artifacts when non-smooth)
- Inf-sup stable mixed finite element method
  - Use discontinuous pressure to enforce local mass conservation
  - Inf-sup constant decays like $\sqrt{\varepsilon}$ for $Q_k - P_{k-1}^{\text{disc}}$
  - Sub-optimal order of accuracy for $Q_k - Q_{k-2}^{\text{disc}}$

Solving saddle-point problems

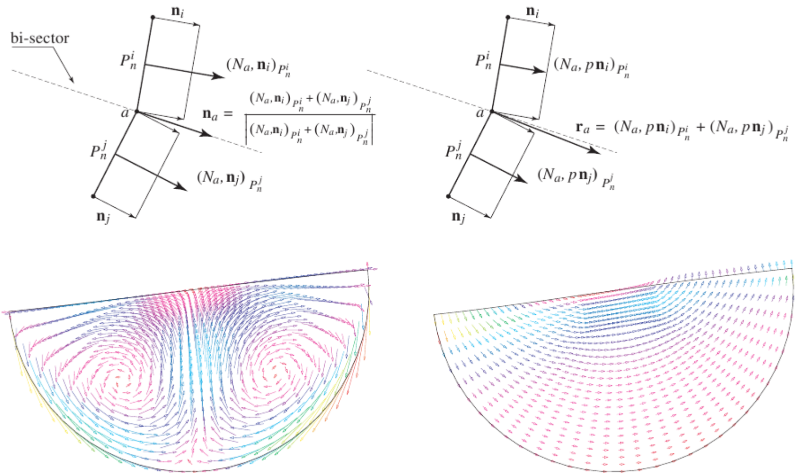
- Not uniformly elliptic: solvers are much less robust
- Standard preconditioners do not work
- Coupled multigrid with Vanka smoothers offer best performance, not robust for stretched grids or anisotropic viscosity
- Block preconditioners require approximate commutators, fragile for strong anisotropy and non-smooth viscosity

Construction of conservative nodal normals

$$n^i = \int_\Gamma \phi^i n$$

- Exact conservation even with rough surfaces
- Definition is robust in 2D and for first-order elements in 3D
- $\int_\Gamma \phi^i = 0$ for corner basis function of undeformed P_2 triangle
- May be negative for sufficiently deformed quadrilaterals
- Mesh motion should use normals from CAD model
  - Difference between CAD normal and conservative normal introduces correction term to conserve mass within the mesh
  - Anomolous velocities if disagreement is large (fast moving mesh, rough surface)
- Normal field not as smooth/accurate as desirable (and achievable with non-conservative normals)
  - Mostly problematic for surface tension
  - Walkley et al, *On calculation of normals in free-surface flow problems*, 2004

Need for well-balancing



(Behr, *On the application of slip boundary condition on curved surfaces*, 2004)

"No" boundary condition

- Integration by parts produces

$$\int_\Gamma v \cdot T\sigma \cdot n, \qquad \sigma = \eta Du - p1, \qquad T = 1 - n \otimes n$$

- Continuous weak form requires either
  - Dirichlet: $u|_\Gamma = f \implies v|_\Gamma = 0$
  - Neumann/Robin: $\sigma \cdot n|_\Gamma = g(u,p)$
- Discrete problem allows integration of $\sigma \cdot n$ "as is"
  - Extends validity of equations to include Γ
  - Not valid for continuum equations
  - Introduced by Papanastasiou, Malamataris, and Ellwood, 1992 for Navier-Stokes outflow boundaries
  - Griffiths, *The 'no boundary condition' outflow boundary condition*, 1997
    - Proves L^∞ order of accuracy $\mathcal{O}((h + 1/\mathrm{Pe})^{p+1})$ for Galerkin finite elements of order p (linear advection-diffusion)
    - Demonstrates equivalence with collocation at Radau points in outflow element
  - Used in slip boundary conditions by Behr 2004

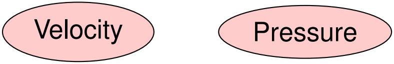
ALE form

After discretization in time ($\alpha \propto 1/\Delta t$) we have a Jacobian

$$\begin{bmatrix} A_{II} & A_{I\Gamma} & & & & \\ & \alpha M_{\Gamma\Gamma} & & -N_{\Gamma\Gamma} & & \\ G_{II} & G_{\Gamma I} & B_{II} & B_{I\Gamma} & C_I^T & D_I \\ G_{I\Gamma} & G_{\Gamma\Gamma} & B_{\Gamma I} & B_{\Gamma\Gamma} & C_\Gamma^T & D_\Gamma \\ G_{Ip} & G_{\Gamma p} & C_I & C_\Gamma & & \\ \alpha E_I & \alpha E_\Gamma & F_I & F_\Gamma & & \alpha M_\Theta + J \end{bmatrix} \begin{bmatrix} x_I \\ x_\Gamma \\ u_I \\ u_\Gamma \\ p \\ \Theta \end{bmatrix}$$

- pseudo-elasticity for mesh motion
- $(\dot{x} - u) \cdot n =$ accumulution
- “just” geometry
- Stokes problem
- temperature dependence of rheology
- convective terms and strain heating in heat transport
- thermal advection-diffusion

Multi-physics coupling in PETSc



- package each "physics" independently
- solve single-physics and coupled problems
- semi-implicit and fully implicit
- reuse residual and Jacobian evaluation unmodified
- direct solvers and efficient fieldsplit without recompilation
- use the best possible matrix format for each physics (symmetric block size 3)
- matrix-free anywhere
- multiple levels of nesting

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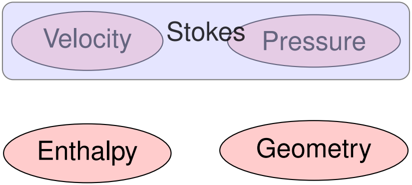

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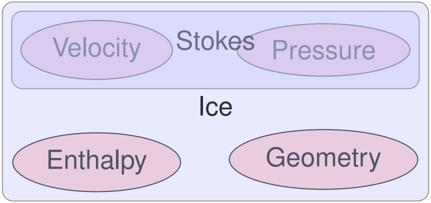

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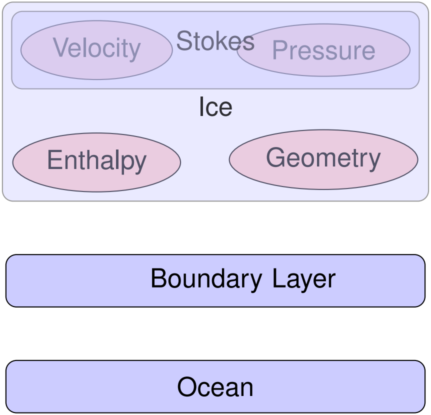

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Outlook

- We have textbook multigrid efficiency for hydrostatic equations
- Technical challenges for Stokes
- Local conservation is critical, well-balanced slip
- Singularities: reentrant corners, transition from frozen
  to slip bounadry conditions, grounded margins, grounding lines
- Stiff geometric coupling terms
- Finally a good algebraic interface for tightly-coupled multiphysics
- IMEX time integration: additive Runge-Kutta

Tools

- PETSc `http://mcs.anl.gov/petsc`
  - ML, Hypre, MUMPS
- ITAPS `http://itaps.org`
  - MOAB, CGM, Lasso