A parallel unstructured implicit 3D polythermal model for outlet glaciers

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Outline

Conservative models

Conforming boundaries and model non-smoothness

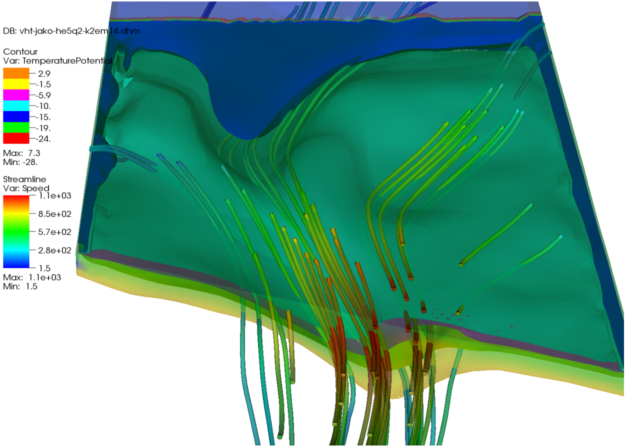
Multilevel Solvers

Bathymetry and stickyness distribution

- Bathymetry:
  - Aspect ratio $\varepsilon = [H]/[x] \ll 1$
  - Need surface *and* bed slopes to be small
- Stickyness distribution:
  - Limiting cases of plug flow versus vertical shear
  - Stress ratio: $\lambda = [\tau_{xz}]/[\tau_{\text{membrane}}]$
  - Discontinuous: frozen to slippery transition at ice stream margins
- Standard approach in glaciology:
  Taylor expand in ε and sometimes λ, drop higher order terms.

  $\lambda \gg 1$ Shallow Ice Approximation (SIA), no horizontal coupling

  $\lambda \ll 1$ Shallow Shelf Approximation (SSA), 2D elliptic solve in map-plane
  - Hydrostatic and various hybrids, 2D or 3D elliptic solves
- Bed slope is discontinuous and of order 1.
  - Taylor expansions no longer valid
  - Numerics require high resolution, subgrid parametrization, short time steps
  - Still get low quality results in the regions of most interest.
- Basal sliding parameters are discontinuous.



DB: vht-jako-he5q2-k2em14.dhm
Contour
Var: TemperaturePotential
2.9
-1.5
-5.9
-10.
-15.
-19.
-24.
Max: 7.3
Min: -28.
Streamline
Var: Speed
1.1e+03
8.5e+02
5.7e+02
2.8e+02
1.5
Max: 1.1e+03
Min: 1.5

Polythermal ice

- Interface tracking methods (e.g. Greve's SICOPOLIS)
  - Different fields for temperate and cold ice.
  - Lagrangian or Eulerian, problems with changing topology
  - No discrete conservation
- Interface capturing
  - Enthalpy: Aschwanden, Bueler, Khroulev, Blatter (J. Glac. 2012/PISM)
    - Not in conservation form
    - Only conservative for infinitesimal melt fraction
  - Energy
    - Conserves mass, momentum, and energy for arbitrary melt fraction
    - Implicit equation of state

Conservative (non-Boussinesq) two-phase ice flow

Find momentum density ρu, pressure p, and total energy density E:

$$(\rho u)_t + \nabla\cdot(\rho u \otimes u - \eta D u_i + p1) - \rho g = 0$$

$$\rho_t + \nabla\cdot \rho u = 0$$

$$E_t + \nabla\cdot\big((E+p)u - k_T \nabla T - k_\omega \nabla \omega\big) - \eta D u_i : D u_i - \rho u \cdot g = 0$$

- Solve for density ρ, ice velocity u_i, temperature T, and melt fraction ω using constitutive relations.
  - Simplified constitutive relations can be solved explicitly.
  - Temperature, moisture, and strain-rate dependent rheology η.
  - High order FEM, typically Q_3 momentum & energy
- DAEs solved implicitly after semidiscretizing in space.
- Preconditioning using nested fieldsplit
- Thermomechanical steady state in about 10 nonlinear iterations

Outline

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Conforming boundaries and model non-smoothness

Multilevel Solvers

Why care about conforming and non-smoothness?

- Subshelf ocean circulation is physically richer and more difficult to solve than ice flow
  - Heat transfer is sensitive to boundary layer processes with thickness $< 1\,\text{m}$, $\mathrm{Re} \sim 10^6$
  - Countless engineering studies: wall modeling is limited, significant normal resolution still necessary
  - Unaligned interface anisotropy is bad for Eulerian AMR methods
- We care about high-dimensional sensitivity analysis and inversion
  - Forward model evaluations alone are an inefficient way to explore a high-dimensional space
  - Each source of model non-smoothness requires a lot of analysis to use adjoint methods
  - Conforming moving mesh methods eliminate all but “essential” non-smoothness (like contact)

Mesh motion via Inverse Beltrami formulation

- nonlinear elliptic (or parabolic) equation for mesh location
- prescribe resolution and anisotropy using target metric tensor
- efficient solution using Newton-Krylov multigrid or nonlinear multigrid
- conservative slip boundary conditions at most surfaces
- ALE transport scheme corrected to satisfy geometric conservation law

Transport

Theorem (Godunov 1954)

Non-oscillatory linear spatial discretizations for transport are at most first order accurate.

- First order accurate discretizations have unacceptably high numerical diffusion
- Discretization choices:
  - Limited or reconstructed finite volume or finite difference
    - Second order TVD limiters have corners
    - Weighted Essential Non-Oscillatory (WENO) smooth, but very nonlinear
    - Central or upwind
    - Inconvenient for unstructured grids
  - Nonlinearly stabilized continuous finite element
    - Currently used, but fragile and messy
  - Discontinuous Galerkin
    - Cell-wise entropy stability without limiters
    - Improved robustness with smooth limiters
    - Transitioning to this

Joke

Outline

Why do we need multilevel solvers?

- Elliptic problems are globally coupled
- Without a coarse level, number of iterations proportional to inverse mesh size
- High-volume local communication is an inefficient way to communicate long-range information, bad for parallel models
- Most important with 3D flow features and/or slippery beds
- Nested/split multilevel methods
  - Decompose problem into simpler sub-problems, use multilevel methods on each
  - Good reuse of existing software
  - More synchronization due to nesting, more suitable after linearization
- Monolithic/coupled multilevel methods
  - Better convergence and lower synchronization, but harder to get right
  - Internal nonlinearities resolved locally
  - More discretization-specific, less software reuse

Relative effect of the blocks

$$J = \begin{pmatrix} J_{uu} & J_{up} & J_{uE} \\ J_{pu} & 0 & 0 \\ J_{Eu} & J_{Ep} & J_{EE} \end{pmatrix}.$$

- J_{uu} Viscous/momentum terms, nearly symmetric, variable coefficionts, anisotropy from Newton.
- J_{up} Weak pressure gradient, viscosity dependence on pressure (small), gravitational contribution (pressure-induced density variation). Large, nearly balanced by gravitational forcing.
- J_{uE} Viscous dependence on energy, very nonlinear, not very large.
- J_{pu} Divergence (mass conservation), nearly equal to J_{up}^T.
- J_{Eu} Sensitivity of energy on momentum, mostly advective transport. Large in boundary layers with large thermal/moisture gradients.
- J_{Ep} Thermal/moisture diffusion due to pressure-melting, $u \cdot \nabla$.
- J_{EE} Advection-diffusion for energy, very nonlinear at small regularization. Advection-dominated except in boundary layers and stagnant ice, often balanced in vertical.

How much nesting?

$$P_1 = \begin{pmatrix} J_{uu} & J_{up} & J_{uE} \\ 0 & B_{pp} & 0 \\ 0 & 0 & J_{EE} \end{pmatrix}$$

- B_{pp} is a mass matrix in the pressure space weighted by inverse of kinematic viscosity.
- Elman, Mihajlović, Wathen, JCP 2011 for non-dimensional isoviscous Boussinesq.
- Works well for non-dimensional problems on the cube, not for realistic parameters.

$$P = \begin{bmatrix} \begin{pmatrix} J_{uu} & J_{up} \\ J_{pu} & 0 \end{pmatrix} & \\ \begin{pmatrix} J_{Eu} & J_{Ep} \end{pmatrix} & J_{EE} \end{bmatrix}$$

- Inexact inner solve using upper-triangular with B_{pp} for Schur.
- Another level of nesting.
- GCR tolerant of inexact inner solves.
- Outer converges in 1 or 2 iterations.

- Low-order preconditioning full-accuracy unassembled high order operator.
- Build these on command line with PETSc `PCFieldSplit`.

Full Approximation Scheme

$$\tilde{u}^h \leftarrow S^h_{\text{pre}} u^h_0 \qquad \text{pre-smooth}$$

$$L^H u^H = I^H_h f^h + \underbrace{L^H \hat{I}^H_h \tilde{u}^h - I^H_h L^h \tilde{u}^h}_{\tau^H_h} \qquad \text{solve coarse problem for } u^H$$

$$u^h \leftarrow S^h_{\text{post}} \left[\tilde{u}^h + I^h_H (u^H - \hat{I}^H_h \tilde{u}^h) \right] \qquad \text{apply correction and post-smooth}$$

- Nonlinearities from spatial discretization fixed locally
- No assembled matrices so better floating point utilization, less memory
- Makes progress on all physical components at once
- FD and DG good, less efficient for continuous finite element methods
- Influence of surface evolution is low rank, no need to visit finest level on each iteration

Outlook

- Basal hydrology model
  - Need mesh-independent statistics
  - Numerical homogenization?
- True inverse and sensitivity support, but need to invert for the right thing
- Dynamic remeshing after large topology changes
- Finish FAS multigrid for full coupled system including geometry
- User-friendliness of process
  - Currently using georeferenced initial/boundary data via GDAL
  - But meshing process is not fully automatic
  - Better: multiresolution database (Mark Fahnestock)