High-order discretizations for 3D ice sheet grounding line dynamics

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The need for models of grounding line dynamics

Accurate prediction of sea level rise is a problem of significant societal relevance and the contributor with greatest uncertainty is overwhelmingly ice sheet dynamics. In one spatial dimension, it is readily shown [6] that the point where ice becomes floating (the grounding line) is unstable on a reverse-sloping bed. Alaska's Columbia Glacier is a present example of this instability, where a long period of quasi-steady behavior is followed by sustained rapid retreat combined with greatly increased discharge.

The leading source of uncertainty on the century time scale comes from West Antarctica, especially the Amundsen Sea sector where Pine Island and Thwaites glaciers harbor enough ice to produce over one meter of effective sea level rise, in a regime widely thought to be unstable according to present theory. However, the theory is not strictly applicable in three dimensions, and even if the instability is still present for the relevant geometry, it does not predict the rate of retreat. To further complicate issues, ocean circulation in this region is very strong, with melt rates on the order of 50 meters per year [5], and time scales much faster than typically associated with ice sheets.

High-order methods

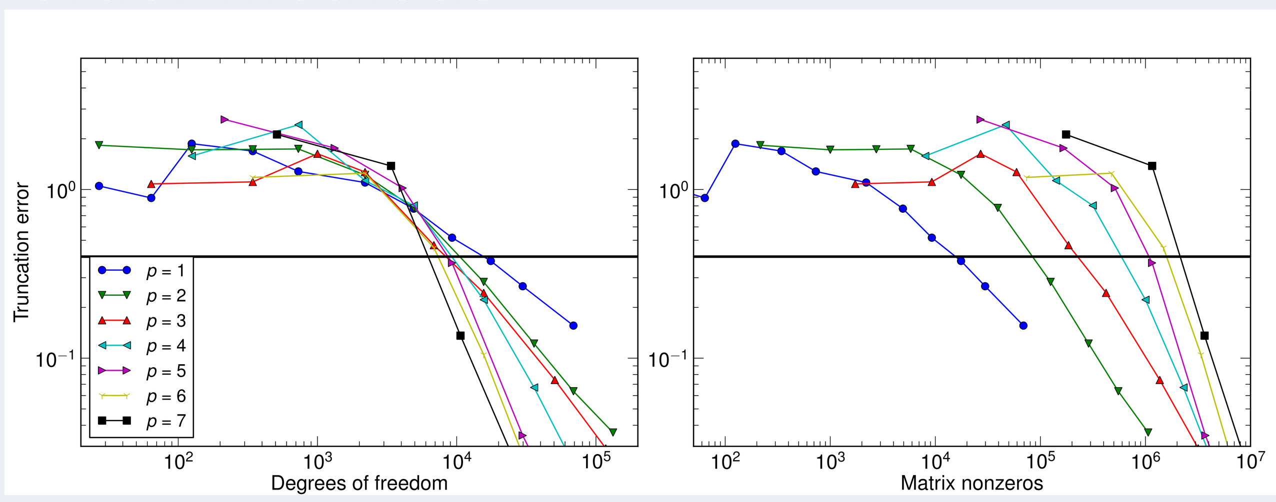
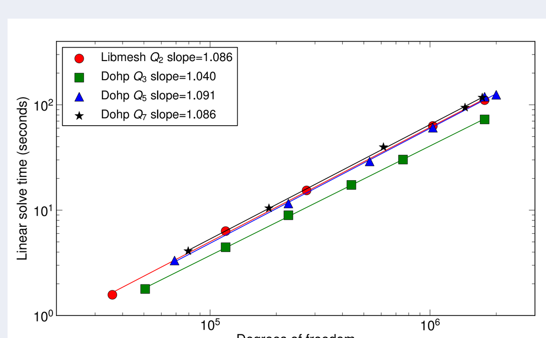


Figure: Two measures of cost for 3D high-order finite element methods. The dark line represents an example suitable accuracy. Finite element methods of order p involve element matrices of size $p^3 \times p^3$. Since solver cost is almost always superlinear in the number of nonzeros in the Jacobian, high-order methods that involve assembling the true Jacobian are rarely economical.

It is not necessary to assemble the true Jacobian in order to solve Newton steps using Krylov methods, however some assembled matrices are required for preconditioning. We use a dual-order scheme [2] to assemble spectrally equivalent, but much sparser matrices. This permits use of very high order schemes with cost similar to conventional quadratic elements.



Efficient representation of Jacobians

- Continuous weak form: find $u \in \mathcal{V}_D$ such that
$$v^T F(u) \sim \int_\Omega v \cdot f_0(u, \nabla u) + \nabla v : f_1(u, \nabla u) = 0 \qquad \forall v \in \mathcal{V}_0$$
- Weak form of the Jacobian
$$v^T J(w) u \sim \int_\Omega \begin{bmatrix} v^T & \nabla v^T \end{bmatrix} \begin{bmatrix} f_{0,0} & f_{0,1} \\ f_{1,0} & f_{1,1} \end{bmatrix} \begin{bmatrix} u \\ \nabla u \end{bmatrix}$$
$$[f_{i,j}] = \begin{bmatrix} \frac{\partial f_0}{\partial u} & \frac{\partial f_0}{\partial \nabla u} \\ \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial \nabla u} \end{bmatrix} (w, \nabla w)$$
- Represent $J(w)$ by storing $[f_{i,j}]$ at quadrature points.
- Frequently much of $[f_{i,j}]$ is computed while evaluating f_i.
  - Inexpensive *taping* for full-accuracy matrix-free Jacobian
  - Code reuse in preconditioner assembly
- The terms in $[f_{i,j}]$ are easy to compute with symbolic math. Possible to automatically generate code.

Non-Newtonian Stokes problems

- Non-shallow ice dynamics requires finding velocity $\boldsymbol{u}$ and pressure p:
$$-\nabla \cdot (\eta D\boldsymbol{u}) + \nabla p - \boldsymbol{f} = 0$$
$$\nabla \cdot \boldsymbol{u} = 0$$
where
$$D\boldsymbol{u} = \tfrac{1}{2}\left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T\right) \qquad \eta(\gamma) = B(\Theta, \dots)(\epsilon + \gamma)^{\frac{\mathfrak{p}-2}{2}}$$
$$\gamma(D\boldsymbol{u}) = \tfrac{1}{2} D\boldsymbol{u} : D\boldsymbol{u} \qquad \mathfrak{p} = 1 + \tfrac{1}{\mathfrak{n}} \approx \tfrac{4}{3}$$
with boundary conditions
$$(\eta D\boldsymbol{u} - p\boldsymbol{1}) \cdot \boldsymbol{n} = \begin{cases} \boldsymbol{0} & \text{free surface} \\ -\rho_w z \boldsymbol{n} & \text{ice-ocean interface} \end{cases}$$
$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{frozen bed}, \Theta < \Theta_0$$
$$\left.\begin{aligned} \boldsymbol{u} \cdot \boldsymbol{n} &= \boldsymbol{g}_{\text{melt}}(T\boldsymbol{u}, \dots) \\ T(\eta D\boldsymbol{u} - p\boldsymbol{1}) \cdot \boldsymbol{n} &= \boldsymbol{g}_{\text{slip}}(T\boldsymbol{u}, \dots) \end{aligned}\right\} \text{nonlinear slip}, \Theta \geq \Theta_0$$
where $T = \boldsymbol{1} - \boldsymbol{n} \otimes \boldsymbol{n}$ projects into the tangent space and slip is defined by
$$\boldsymbol{g}_{\text{slip}}(T\boldsymbol{u}) = \beta_{\mathfrak{m}}(\dots)|T\boldsymbol{u}|^{\mathfrak{m}-1} T\boldsymbol{u}$$
$$\text{Navier} \quad \mathfrak{m} = 1, \qquad \text{Weertman} \quad \mathfrak{m} \approx \frac{1}{3}, \qquad \text{Coulomb} \quad \mathfrak{m} = 0.$$
- This system involves no time derivatives so the semidiscrete form represents an algebraic constraint to the DAE formulated below. The weak form of the velocity block of the Newton step is: find $(\boldsymbol{u}, p)$ such that
$$\int_\Omega D\boldsymbol{v} : \left[\eta \boldsymbol{1} + \eta' D\boldsymbol{w} \otimes D\boldsymbol{w}\right] : D\boldsymbol{u} - p \nabla \cdot \boldsymbol{v} - q \nabla \cdot \boldsymbol{u} = -\boldsymbol{v} \cdot F(\boldsymbol{w}) \qquad \forall (\boldsymbol{v}, q)$$
which has the indefinite matrix form
$$\begin{bmatrix} A(\boldsymbol{w}) & B^T \\ B & \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = -\begin{bmatrix} F_u(\boldsymbol{w}) \\ 0 \end{bmatrix}.$$
Standard preconditioners perform poorly on such problems so we use incomplete factorization with the Schur complement approximated via approximate commutators, similar to [4].

Arbitrary Lagrange-Eulerian formulation

- pseudo-elasticity for mesh location $\boldsymbol{x}$, displacement $\boldsymbol{w} = \boldsymbol{x} - \boldsymbol{x}_0$
$$-\nabla \cdot \boldsymbol{\sigma} = 0 \qquad \boldsymbol{\sigma} = \mu\left[2D\boldsymbol{w} + (\nabla \boldsymbol{w})^T \nabla \boldsymbol{w}\right] + \lambda \operatorname{tr}(\nabla \boldsymbol{w}) \boldsymbol{1}$$
surface kinematic condition: $(\dot{\boldsymbol{x}} - \boldsymbol{u}) \cdot \boldsymbol{n} = q_{BL}, \; T\boldsymbol{\sigma} \cdot \boldsymbol{n} = 0$
- heat transport: Θ (enthalpy)
$$\underbrace{\rho\left[\frac{\partial}{\partial t}\Theta + (\boldsymbol{u} - \dot{\boldsymbol{x}}) \cdot \nabla \Theta\right]}_{\text{ALE advection}} - \underbrace{\nabla \cdot \left[\kappa(\Theta) \nabla \Theta\right.}_{\text{Fourier/Fick diffusion}} + \underbrace{\left.\boldsymbol{q}_D(\Theta)\right]}_{\text{Darcy flow}} - \underbrace{\eta D\boldsymbol{u} : D\boldsymbol{u}}_{\text{Strain heating}} = 0$$
  - $\kappa(\Theta)$ and $\boldsymbol{q}_D(\Theta)$ are very sensitive near $\Theta = \Theta_0$
- formulation as a DAE

  The fully coupled system is semidiscretized in space, resulting in a differential-algebraic system of the form
$$F(t, X, \dot{X}) = 0$$
  where X is a multivector containing

u	velocity	algebraic
p	pressure	algebraic
x	mesh location	algebraic in domain, differential at surfaces
Θ	enthalpy	differential

General Linear methods for DAE and stiff ODE

- Common integrators such as linear multistep (LMS) and Runge-Kutta (RK) methods have deficiencies for stiff problems
  - Dahlquist's second barrier: A-stable LMS have order at most 2
  - Diagonally implicit RK have stage order 1
  - Singly implicit RK with abscissas outside the step and poor error constants
  - Fully implicit RK require solving a much larger system
  - Lack of robust error estimates
- General linear methods have recently been developed which avoid many of these barriers by offering
  - diagonally implicit computational structure
  - A- and L-stable
  - stage order equal to conventional order
  - asymptotically correct error estimates for the current method *and* methods of order one higher than the current method
- General linear methods for the DAE
$$f(t, x, \dot{x}) = 0$$
  can be written in a tableau similar to Runge-Kutta methods:
$$\begin{bmatrix} Y \\ X^{n+1} \end{bmatrix} = \begin{bmatrix} A & U \\ B & V \end{bmatrix} \begin{bmatrix} h\dot{Y} \\ X^n \end{bmatrix}$$
  - stage values $Y = \{y_1, \dots, y_s\}$ computed sequentially be solving $f(t_i^n, y_i, \dot{y}_i) = 0$ with $\dot{y}_i$ written in terms of y_i using the top block
  - Nordsieck vector passed between steps
$$X = \{x_1, \dots, x_r\} = \{x, h\dot{x}, h^2\ddot{x}, \dots\}$$
- These methods were implemented for DAE in the TS component of PETSc [1].
  - orders 1 to 5, extensible by providing the entries of the tableau $\begin{smallmatrix} A & U \\ B & V \end{smallmatrix}$
  - error estimators and robust step-size adjustment from [3]
  - adaptive-order adaptive-step controller, plugin architecture for extending

Discussion

- efficient high-order spatial discretizations have been applied to each subproblem
- the ALE formulation
  - makes no shallowness assumptions
  - removes accuracy-limiting features of Eulerian schemes
  - naturally admits complex boundary conditions including slip
  - is a "method of lines" discretization suitable for steady-state, bifurcation, and sensitivity analysis, as well as data assimilation
- general linear DAE integrators have several desirable properties that are not possible with more conventional methods

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