Indefinite and physics-based preconditioning

Jed Brown

VAW, ETH Zürich

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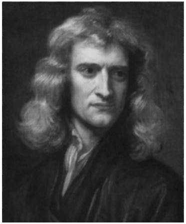
Newton iteration



- Standard form of a nonlinear system

$$F(u) = 0$$

- Iteration

$$\begin{aligned} \text{Solve:} \quad & J(\tilde{u})u = -F(\tilde{u}) \\ \text{Update:} \quad & \tilde{u}_+ \leftarrow \tilde{u} + u \end{aligned}$$

Example (p-Bratu)

Suppose F is a discretization of

$$-\nabla \cdot (\eta \nabla u) - \lambda e^u - f = 0$$

$$\eta(\gamma) = (\frac{\epsilon^2 + \gamma}{\gamma_0})^{\frac{p-2}{2}}, \qquad \gamma = \frac{1}{2} |\nabla u|^2 .$$

Then $J(\tilde{u})u$ is a discretization of

$$-\nabla \cdot (\eta \nabla u + \eta'(\nabla \tilde{u} \cdot \nabla u)\nabla \tilde{u}) - \lambda e^{\tilde{u}} u.$$

Matrices and Preconditioners

Definition (Matrix)

A matrix is a linear transformation between finite dimensional vector spaces.

Definition (Forming a matrix)

Forming or assembling a matrix means defining it's action in terms of entries (usually stored in a sparse format).

Left preconditioning in a Krylov iteration

$$(P^{-1}A)x = P^{-1}b$$
$$\{P^{-1}b, (P^{-1}A)P^{-1}b, (P^{-1}A)^2 P^{-1}b, \dots\}$$

Definition (Preconditioner)

A *preconditioner* $\mathcal{P}$ is a method for constructing a matrix (just a linear function, not assembled!) $P^{-1} = \mathcal{P}(A, A_p)$ using a matrix A and extra information A_p, such that the spectrum of $P^{-1}A$ (or AP^{-1}) is well-behaved.

Domain decomposition

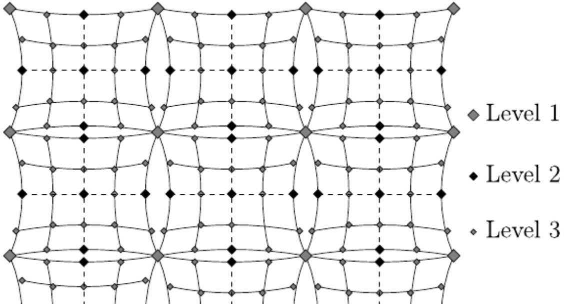
Domain size L, subdomain size H, element size h

Overlapping/Schwarz

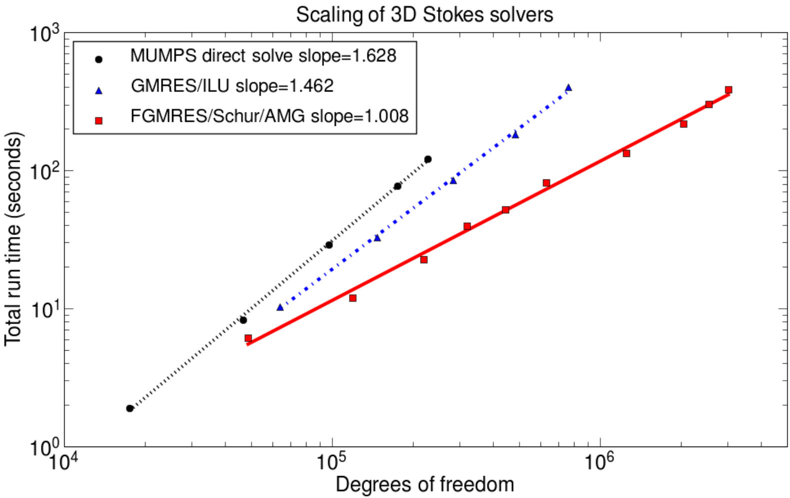
- Solve Dirichlet problems on overlapping subdomains
- No overlap: $its \in \mathcal{O}\left(\frac{L}{\sqrt{Hh}}\right)$,   Overlap: $its \in \mathcal{O}\left(\frac{L}{H}\right)$

BDDC and FETI-DP

- Neumann problems on subdomains with coarse grid correction
- $its \in \mathcal{O}\left(1 + \log \frac{H}{h}\right)$



Normal preconditioners fail for indefinite problems



Model problem: Stokes system

- Strong form: Find $(u, p) \in V' \times P'$ such that

$$-\eta \nabla^2 u + \nabla p = f$$
$$\nabla \cdot u = 0$$

- Minimization form: Find $u \in V$ which minimizes

$$I(u) = \int_\Omega \eta \nabla u : \nabla u - f \cdot u$$

subject to

$$\nabla \cdot u = 0$$

- Lagrangian:

$$L(u, p) = \int_\Omega \eta \nabla u : \nabla u - p \nabla \cdot u - f \cdot u$$

- Weak form: Find $(u, p) \in V \times P$ such that

$$\int_\Omega \eta \nabla v : \nabla u - q \nabla \cdot u - p \nabla \cdot v - f \cdot v = 0$$

for all $(v, q) \in V' \times P'$.

Stokes

Weak form

Find $(u, p) \in V \times P$ such that

$$\int_\Omega \eta \nabla v : \nabla u - q \nabla \cdot u - p \nabla \cdot v - f \cdot v = 0$$

for all $(v, q) \in V \times P$.

Matrix

$$Jx = \begin{bmatrix} A & B^T \\ B & \end{bmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

Block factorization

$$\begin{bmatrix} A & B^T \\ B & \end{bmatrix} = \begin{bmatrix} 1 & \\ BA^{-1} & 1 \end{bmatrix} \begin{bmatrix} A & B^T \\ & S \end{bmatrix} = \begin{bmatrix} A & \\ B & S \end{bmatrix} \begin{bmatrix} 1 & A^{-1}B^T \\ & 1 \end{bmatrix}$$

where the Schur complement is

$$S = -BA^{-1}B^T.$$

Block factorization

$$\begin{bmatrix} A & B^T \\ B & \end{bmatrix} = \begin{bmatrix} 1 & \\ BA^{-1} & 1 \end{bmatrix} \begin{bmatrix} A & B^T \\ & S \end{bmatrix} = \begin{bmatrix} A & \\ B & S \end{bmatrix} \begin{bmatrix} 1 & A^{-1}B^T \\ & 1 \end{bmatrix}$$

where

$$S = -BA^{-1}B^T.$$

- S is symmetric negative definite if A is SPD and B has full rank.
- S is dense
- We only need to multiply B, B^T with vectors.
- We need a preconditioner for A and S.
- Any definite preconditioner (from last time) can be used for A.
- It's not obvious how to precondition S, more on that later.

Reduced factorizations are sufficient

Theorem (GMRES convergence)

GMRES applied to

$$Tx = b$$

converges in n steps for all right hand sides if there exists a polynomial of degree n such that $p_n(A) = 0$ and $p_n(0) = 1$. That is, if the minimum polynomial of A has degree n.

A lower-triangular preconditioner

Left precondition J:

$$T = P^{-1}J = \begin{bmatrix} A & \\ B & S \end{bmatrix}^{-1} \begin{bmatrix} A & B^T \\ B & \end{bmatrix}$$

$$= \begin{bmatrix} A^{-1} & \\ -S^{-1}BA^{-1} & S^{-1} \end{bmatrix} \begin{bmatrix} A & B^T \\ B & \end{bmatrix} = \begin{bmatrix} 1 & A^{-1}B^T \\ & 1 \end{bmatrix}$$

Since $(T-1)^2 = 0$, GMRES converges in at most 2 steps.

Preserving symmetry, for CG or MinRes

Either $P^{-1}A$ must be symmetric or both P^{-1} and A must be symmetric

$$P^{-1} = \begin{bmatrix} A & \\ & -S \end{bmatrix}^{-1}$$

$$T = P^{-1}J = \begin{bmatrix} A^{-1} & \\ & -S^{-1} \end{bmatrix} \begin{bmatrix} A & B^T \\ B & \end{bmatrix} = \begin{bmatrix} 1 & A^{-1}B^T \\ -S^{-1}B & \end{bmatrix}$$

$$\left(T - \frac{1}{2}\right)^2 = \begin{bmatrix} \frac{1}{4} - A^{-1}B^TS^{-1}B & \\ & \frac{5}{4} \end{bmatrix}$$

$$\left(T - \frac{1}{2}\right)^2 - \frac{1}{4} = \begin{bmatrix} -A^{-1}B^TS^{-1}B & \\ & 1 \end{bmatrix}$$

Now $Q = -A^{-1}B^TS^{-1}B$ is a projector ($Q^2 = Q$) so

$$\left[\left(T - \frac{1}{2}\right)^2 - \frac{1}{4}\right]^2 = \left(T - \frac{1}{2}\right)^2 - \frac{1}{4}$$

Rearranging, $T(T-1)(T^2 - T - 1) = 0$. GMRES converges in at most 3 iterations.

Preconitioning the Schur complement

- $S = -BA^{-1}B^T$ is dense so we can't form it, but we need S^{-1}.

Least squares commutator

Suppose B is square and nonsingular. Then

$$S^{-1} = -B^{-T}AB^{-1}.$$

B is not square, replace B^{-1} with Moore-Penrose pseudoinverse

$$B^\dagger = B^T(BB^T)^{-1}, \qquad (B^T)^\dagger = (BB^T)^{-1}B.$$

Then

$$\hat{S}^{-1} = -(BB^T)^{-1}BAB^T(BB^T)^{-1}.$$

- Navier-Stokes: $-BB^T \sim (\nabla \cdot)\nabla$ is essentially a homogeneous Poisson operator in the pressure space
- Multigrid on BB^T is an effective preconditioner.
- Requires 2 Poisson preconditioners per iteration

Preconditioning the Schur complement

Physics-based commutators

- Unsteady Navier-Stokes with Picard linearization:

$$A \sim \left(\frac{1}{\alpha} - \eta\nabla^2 + w\cdot\nabla\right) \qquad B \sim -(\nabla\cdot)$$

- Suppose we have formal commutativity

$$-\nabla\cdot\left(\frac{1}{\alpha} - \eta\nabla^2 + w\cdot\nabla\right)^{-1}\nabla = (-\nabla^2)\left(\frac{1}{\alpha} - \eta\nabla^2 + w\cdot\nabla\right)^{-1}$$

- Discrete form

$$S = -BA^{-1}B^T \approx (-BM_u^{-1}B^T)A_p^{-1}M_p = \hat{S}$$

where M_u, M_p are mass matrices, $L = -BM_u^{-1}B^T$ a homogeneous Laplacian in the pressure space. Then

$$\hat{S}^{-1} = M_p^{-1}A_pL^{-1}$$

- Stokes: $S \sim (-\nabla^2)(-\eta\nabla^2)^{-1} \sim \frac{1}{\eta}$, mass matrix scaled by $\frac{1}{\eta}$.

Physics-based preconditioners

Shallow water with stiff gravity wave

h is hydrostatic pressure, u is velocity, $\sqrt{gh}$ is fast wave speed

$$h_t - (uh)_x = 0$$
$$(uh)_t + (u^2 h)_x + ghh_x = 0$$

Semi-implicit method

Suppress spatial discretization, discretize in time, implicitly for the terms contributing to the gravity wave

$$\frac{h^{n+1} - h^n}{\Delta t} + (uh)_x^{n+1} = 0$$
$$\frac{(uh)^{n+1} - (uh)^n}{\Delta t} + (u^2 h)_x^n + gh^n h_x^{n+1} = 0$$

Rearrange, eliminating $(uh)^{n+1}$

$$\frac{h^{n+1} - h^n}{\Delta t} - \Delta t (gh^n h_x^{n+1})_x = -S_x^n$$

Delta form

- Preconditioner should work like the Newton step

$$-F(x) \mapsto \delta x$$

- Recast semi-implicit method in delta form

$$\frac{\delta h}{\Delta t} + (\delta uh)_x = -F_0$$
$$\frac{\delta uh}{\Delta t} + gh^n(\delta h)_x = -F_1$$

- Eliminate δuh

$$\frac{\delta h}{\Delta t} - \Delta t(gh^n(\delta h)_x)_x = -F_0 + (\Delta t F_1)_x$$

- Solve for δh, then evaluate

$$\delta uh = -\Delta t\left[gh^n(\delta h)_x - F_1\right]$$

- Fully implicit solver
  - Is nonlinearly consistent (no splitting error)
  - Can be high-order in time
  - Leverages existing code (the semi-implicit method)